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On the energy and momentum of an accelerated charged particle and the sources of radiation

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Abstract

We give a systematic development of the theory of the radiation field of an accelerated charged particle with reference to an inertial reference frame in flat spacetime. Special emphasis is given to the role of the Schott energy and momentum in the energy–momentum balance of the charge and its field. It is shown that the energy of the radiation field does not, in general, come from the work performed upon the charge by an external force. The radiation energy also receives a contribution from a conversion of the velocity dependent near-field to the acceleration dependent radiation field. We also exhibit the role of momentum conservation in connection with a radiating electric charge and its electromagnetic field.

1. Introduction

In terms of classical (non-quantum) electrodynamics we shall give a systematic treatment showing how energy and momentum are stored and transformed when a charged particle is performing an arbitrary, accelerated motion. Our main points are (i) to give a clear presentation of the general theory and (ii) to demonstrate the significance of the Schott energy and momentum in the energy and momentum budget of a radiating charge. In particular, we shall show how transformation between Scott energy–momentum and radiation field energy–momentum takes place in the case where the charge performs a uniformly accelerated motion. The problems about radiation from a charged particle performing this type of motion have been discussed by several authors for a long period of time [1–10]. The special case where the charge moves with constant velocity along a curved, for example circular motion, path is also considered. In these cases the radiated energy is supported in quite different ways.

1.1. The general theory of an accelerated charge and its electromagnetic field

The particle produces a field propagating on its future light cone. The field tensor $F^{\mu\nu}$ is written as the sum of a generalized Coulomb field, $F_I^{\mu\nu}$, and a radiation field, $F_{II}^{\mu\nu}$,

$$F^{\mu\nu} = F_I^{\mu\nu} + F_{II}^{\mu\nu}. \tag{1}$$

The field $F_I^{\mu\nu}$ is independent of the acceleration, and $F_{II}^{\mu\nu}$ is of first order in the acceleration.

Inserting equation (1) in the expression of the energy–momentum tensor for an electromagnetic field,

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F_\alpha^\mu F^{\nu\alpha} - \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) \tag{2}$$

we get, in Teitelboim's notation [11]

$$T^{\mu\nu} = T_I^{\mu\nu} + T_{II}^{\mu\nu} \tag{3}$$

where $T_{II}^{\mu\nu}$ is the energy–momentum tensor of the radiation field, $F_{II}^{\mu\nu}$, and $T_I^{\mu\nu}$ is the energy–momentum tensor of the generalized Coulomb field, $F_I^{\mu\nu}$, in addition to the cross terms between the two fields. Teitelboim showed that $\partial T_I^{\mu\nu}/\partial x^\mu = 0$ and $\partial T_{II}^{\mu\nu}/\partial x^\mu = 0$ everywhere outside the worldline of the particle.

A pleasant property of $T_I^{\mu\nu}$ (see [12]) is that the volume integral of $T_I^{0\nu}$ taken at a fixed point of time over the whole 3-space outside the particle (considered as a sphere of vanishing small radius) is given by the particle's velocity and acceleration at the same point of time. Hence, the resulting 4-momentum, P_I^ν, is a state function of the particle although it represents the energy–momentum of type I of the particle's field. This 4-momentum is found to be

$$P_I^\nu = \left(P_I^0, \vec{P}_I\right) = m_0 U^\nu - \frac{2}{3c^3}Q^2 A^\nu \tag{4}$$

where Q is the charge of the particle, $U^\nu = \gamma(c, \vec{v})$, $\gamma = 1/\sqrt{1 - v^2/c^2}$ its 4-velocity and A^ν its 4-acceleration,

$$A^\nu = (A^0, \vec{A}) = \frac{\mathrm{d}U^\nu}{\mathrm{d}\tau} = \gamma^2\left(\gamma^2\frac{\vec{v}}{c}\cdot\vec{a}, \vec{a} + \gamma^2\left(\frac{\vec{v}}{c}\cdot\vec{a}\right)\frac{\vec{v}}{c}\right) = \left(\frac{\vec{v}}{c}\cdot\vec{A}, \vec{A}\right). \tag{5}$$

The physical rest mass m_0 in equation (4) is renormalized from the electromagnetic mass (divergent for a point particle) and a possible non-electromagnetic contribution.

The last term in equation (4) is the Schott [13, 14] 4-momentum,

$$P_S^\nu = -\frac{2}{3c^3}Q^2 A^\nu. \tag{6}$$

The Schott energy is

$$cP_S^0 = -\frac{2}{3c^2}Q^2 A^0 = -\frac{2}{3c^3}Q^2\gamma^4\vec{v}\cdot\vec{a}. \tag{7}$$

Note that the system has an energy and a momentum not only due to the rest mass and velocity, but also due to the acceleration. The Schott energy P_S^0 is an acceleration energy [15–17], and $\vec{P}_S$ an acceleration momentum [12], and both come from the field of the particle.

As opposed to $\vec{P}_I^\nu$ the radiation 4-momentum $\vec{P}_{II}^\nu$ depends upon the whole prehistory of the particle. According to the version of Larmor's relativistic formula valid in inertial reference frames, the radiated 4-momentum per proper time is

$$\frac{\mathrm{d}P_{II}^\nu}{\mathrm{d}\tau} = \frac{2}{3c^5}Q^2 g^2 U^\nu = \mathbb{R}U^\nu \tag{8}$$

where $g = (A_\mu A^\mu)^{1/2}$ is the proper acceleration and $\mathbb{R}c^2 = (2/3c^3)Q^2 g^2$ is the radiated power.

Since the emitted energy and momentum are conserved during the motion away from the charge, the 4-momentum in the radiation field at the point of time T may be expressed as

$$P_{II}^{\nu} = \int_{-\infty}^{\tau(T)} \mathbb{R} U^{\nu} \, \mathrm{d}\tau \tag{9}$$

and the total 4-momentum of particle and field becomes

$$P^{\nu} = P_I^{\nu} + P_{II}^{\nu} = m_0 U^{\nu} + P_S^{\nu} + P_{II}^{\nu}. \tag{10}$$

When the particle is free, P^{ν} is constant. If an external force, F^{ν}, is acting we get

$$F^{\nu} = \dot{P}^{\nu} = m_0 A^{\nu} + \dot{P}_S^{\nu} + \mathbb{R} U^{\nu} \tag{11}$$

where the dot denotes differentiation with respect to the proper time τ of the particle. This is the Lorentz–Dirac equation of motion (in our notation). From the scalar products $U_{\nu} U^{\nu} = -c^2$, $U_{\nu} A^{\nu} = 0$, $U_{\nu} \dot{A}^{\nu} = -g^2$ it follows that $U_{\nu} F^{\nu} = 0$. This means that F^{ν} may be expressed as

$$F^{\nu} = \gamma \left(\frac{\vec{v}}{c} \cdot \vec{F}, \vec{F} \right) \tag{12}$$

where $\vec{F}$ is a 3-force, and F^0 is the work per unit proper time. Note that it is not possible to express $\mathbb{R} U^{\nu}$ (or $\dot{P}_S^{\nu}$) in the form (12). This means that the amounts of radiated energy and momentum cannot be fully accounted for by one single force acting upon the particle.

According to equation (8) $\mathbb{R}\vec{v}$ is the radiated momentum per unit laboratory time. A force of this magnitude acting upon the particle produces a power $\mathbb{R} v^2$ which is different from the radiated power $\mathbb{R} c^2$. In two examples below we shall see how the Schott energy and Schott momentum take part in the radiation process.

The equation of motion (11) is usually written as

$$F^{\nu} = m_0 A^{\nu} - \Gamma^{\nu} \tag{13}$$

where Γ^{ν} is the Abraham vector

$$\Gamma^{\nu} = -\dot{P}_S^{\nu} - \mathbb{R} U^{\nu}. \tag{14}$$

It may be written as

$$\Gamma^{\nu} = \gamma \left(\frac{\vec{v}}{c} \cdot \vec{\Gamma}, \vec{\Gamma} \right) \tag{15}$$

where $\vec{\Gamma}$ is *the field reaction force*,

$$\vec{\Gamma} = -\frac{\mathrm{d}\vec{P}_S}{\mathrm{d}T} - \mathbb{R}\vec{v} \tag{16a}$$

and $-\mathbb{R}\vec{v}$ is the *radiation reaction force* [18] which always acts against the motion. The power due to the Abraham vector is

$$\vec{v} \cdot \vec{\Gamma} = \frac{c}{\gamma} \Gamma^0 = -\frac{c \, \mathrm{d}P_S^0}{\mathrm{d}T} - \mathbb{R} c^2. \tag{16b}$$

From equations (13) and (16) one gets

$$\vec{F} = \frac{1}{\gamma} m_0 \vec{A} - \vec{\Gamma} = m_0 \frac{\mathrm{d}\vec{U}}{\mathrm{d}T} + \frac{\mathrm{d}\vec{P}_S}{\mathrm{d}T} + \mathbb{R}\vec{v} \tag{17a}$$

$$\vec{v} \cdot \vec{F} = \frac{c}{\gamma} m_0 A^0 - \frac{c}{\gamma} \Gamma^0 = m_0 \frac{c \, \mathrm{d}U^0}{\mathrm{d}T} + \frac{c \, \mathrm{d}P_S^0}{\mathrm{d}T} + \mathbb{R} c^2 \tag{17b}$$

where $\vec{P}_S$ is the Schott momentum and cP_S^0 the Schott energy. Hence the power provided by the external force is equal to the change of kinetic energy of the charge plus the change of Schott energy per unit time plus the radiated power. It is tempting to conclude that the left-hand side of equation (17*b*) and the first term on the right-hand side vanish for circular motion with constant speed, so that the radiated energy comes from the Schott energy. Below we shall show that this is not the case.

2. Some special types of motion

In this section we shall consider motions where $\Gamma^\nu = 0$ and motions where $A^0 = 0$. In the first case the radiation energy comes from the Schott energy. In the second case the Schott energy is zero.

(i) $\Gamma^\nu = 0$

Putting $\Gamma^\nu = 0$ in equations (13) and (14) we get

$$\mathbb{R}U^\nu = -\dot{P}_S^\nu \tag{18}$$

and

$$F^\nu = m_0 A^\nu. \tag{19}$$

The first equation shows that the radiated 4-momentum comes from the Schott 4-momentum. That is, the sum of the 4-momentum in the radiation field and the Schott 4-momentum is constant. According to equation (19) the external force is not engaged in the radiation in this case. The charge moves as if it is neutral.

Using equation (6), equation (18) may be expressed as

$$A_\mu A^\mu U^\nu = \dot{A}^\nu c^2. \tag{20}$$

Taking the scalar product with A^ν we get $A_\nu \dot{A}^\nu = 0$, i.e.

$$A_\nu A^\nu = \text{constant} \tag{21}$$

$$\mathbb{R} = \text{constant}. \tag{22}$$

Thus, in a motion where $\Gamma^\nu = 0$, the proper acceleration $g = (A_\nu A^\nu)^{1/2}$ is constant, and the particle radiates energy at a constant rate. The Schott energy decreases at the same rate. Equation (20) may now be written as

$$g^2 U^\nu = c^2 \frac{\mathrm{d}^2 U^\nu}{\mathrm{d}\tau^2} \tag{23}$$

where g is constant.

The equation has the following solutions:

(a) Particle at rest or moving along a straight line with constant velocity ($g = 0$).
(b) Hyperbolic motion and Lorentz transforms of this.

In the case of hyperbolic motion along the X-axis

$$X = \frac{c^2}{g}\cosh\frac{g\tau}{c}, \qquad T = \frac{c}{g}\sinh\frac{g\tau}{c} \tag{24}$$

which gives

$$U^\nu = \frac{g}{c}(X, cT, 0, 0), \tag{25a}$$

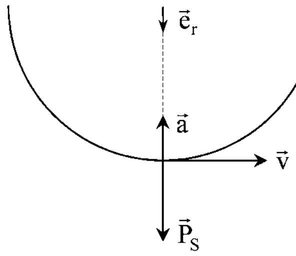

Figure 1. Charged particle in circular motion with constant velocity v. $\vec{P}_S$ is the Schott momentum.

$$A^\nu = \frac{g^2}{c^2}(cT, X, 0, 0), \tag{25b}$$

$$\dot{A}^\nu = \frac{g^3}{c^3}(X, cT, 0, 0). \tag{25c}$$

It follows that

$$\frac{2}{3c^3}Q^2\dot{A}^\nu = \mathbb{R}U^\nu \tag{26}$$

which confirms that the Abraham vector $\Gamma^\nu = 0$.

(ii) $A^0 = 0$

Putting $A^0 = 0$ in equation (5) we find $\mathrm{d}U^0/\mathrm{d}\tau = 0$ and $\vec{v}\cdot\vec{a} = 0$, i.e. γ and v are constants and $\vec{a}\perp\vec{v}$. In this case the 4-acceleration may be written as

$$A^\nu = (0, \gamma^2\vec{a}). \tag{27}$$

From equations (6) and (7) we then get the Schott energy

$$cP_S^0 = -\frac{2}{3c^2}Q^2A^0 = 0 \tag{28a}$$

and the Schott momentum

$$\vec{P}_S = -\frac{2}{3c^3}Q^2\vec{A} = -\frac{2}{3c^3}Q^2\gamma^2\vec{a}. \tag{28b}$$

According to equation (17) the external force is

$$\vec{F} = \gamma m_0\vec{a} + \frac{\mathrm{d}\vec{P}_S}{\mathrm{d}T} + \mathbb{R}\vec{v} \tag{29a}$$

$$\vec{v}\cdot\vec{F} = \mathbb{R}c^2. \tag{29b}$$

The last equation shows that the radiated energy is supported by the tangential component of the external force. The radiated energy per unit time is equal to the power provided by this force.

However, the radiated momentum is not due only to this force. In order to see this most clearly we consider circular motion with radius r and constant speed v (see figure 1).

In this case the Schott 4-momentum is

$$\vec{P}_S = -\frac{2}{3c^3}Q^2\gamma^2\vec{a} = \frac{2}{3c^3}Q^2\gamma^2\frac{v^2}{r}\vec{e}_r \tag{30}$$

where $\vec{e}_r$ is the unit vector in the radial direction. Its rate of change with respect to the laboratory time is given by $\mathrm{d}\vec{e}/\mathrm{d}T = \vec{v}/r$, which gives

$$\frac{\mathrm{d}\vec{P}_S}{\mathrm{d}T} = \frac{2}{3c^3}Q^2\gamma^2\frac{v^2}{r^2}\vec{v}. \tag{31}$$

Putting $\vec{F} = \vec{F}_\parallel + \vec{F}_\perp$ where $\vec{F}_\parallel$ and $\vec{F}_\perp$ are the components of $\vec{F}$ along and orthogonal to $\vec{v}$, we get from equations (29a) and (31)

$$\vec{F}_\perp = \gamma m_0 \vec{a} \tag{32a}$$

$$\vec{F}_\parallel = \frac{\mathrm{d}\vec{P}_S}{\mathrm{d}T} + \mathbb{R}\vec{v} \tag{32b}$$

and further from the earlier result in equation (29b),

$$\vec{v}\cdot\vec{F}_\parallel = \mathbb{R}c^2. \tag{32c}$$

For an uncharged particle the centripetal force $\vec{F}_\perp$ is the only force. In the case of a charged particle a tangential force $\vec{F}_\parallel$ is necessary to keep the velocity constant. The radiated energy comes from the work performed by this force. The radiated momentum is partly due to $\vec{F}_\parallel$ and partly due to the change of direction of the Schott momentum vector.

3. Conclusion

The theory of electromagnetism contains a somewhat hidden ingredient which is essential in obtaining an understanding of the energy–momentum budget of a radiating charge and its field: the Schott energy–momentum which is part of the energy–momentum in the field co-moving with the charge.

This is seen very clearly in connection with a uniformly accelerated charge. Then the field reaction force vanishes. Hence a charged and a neutral particle with equal mass acted upon by equal forces, move beside each other and gain the same kinetic energy. But the charge radiates and the neutral particle does not. So, where does the radiation energy come from? The answer is that part of its Schott energy is transformed to radiation energy.

We have also seen that there exist motions where the particle radiates although the Schott energy is constant. This is the case for curved motion with constant speed.

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